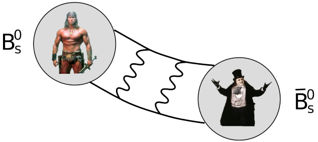
B_s effective lifetimes:

phenomenology with a non-zero decay width difference

Rob Knegjens

B_s decay width difference



"Normal" Modes

$\longrightarrow$

$$\Delta\Gamma_s \equiv \Gamma_{\rm L} - \Gamma_{\rm H}$$

or

$$\boldsymbol{y_s} \equiv \frac{\Gamma_{\rm L} - \Gamma_{\rm H}}{\Gamma_{\rm L} + \Gamma_{\rm H}}$$

Standard Model:

$$\boldsymbol{y_s}|_{\rm SM} = \begin{cases} 0.067 \pm 0.016 \\ 0.074 \pm 0.007 \end{cases}$$

A. Lenz, U. Nierste, arXiv:1102.4274

L. Silvestrini, Beauty 2013

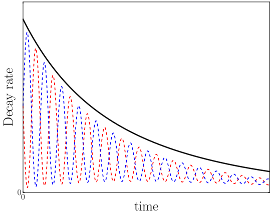
LHCb $B_s \to J/\psi\,\phi$ analysis : $\boldsymbol{y_s}|_{\rm LHCb} = 0.075 \pm 0.012$

LHCb collaboration, arXiv:1304.2600

Untagged decay rates

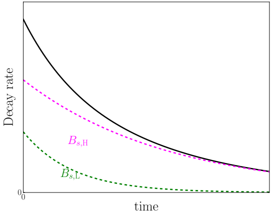
$$\langle \Gamma(B_s(t) \to f) \rangle = \frac{1}{N_{B_s}} \frac{dN_e(B_s \to f)}{dt} = \ldots$$

Flavour basis



$$\Gamma(B_s^0(t) \to f) + \Gamma(\bar{B}_s^0(t) \to f)$$

Mass e-state basis



$$\Gamma(B_{s,\mathrm{H}} \to f)\, e^{-\Gamma_{\mathbf{H}} t} + \Gamma(B_{s,\mathrm{L}} \to f)\, e^{-\Gamma_{\mathbf{L}} t}$$

$$\langle \Gamma_f \rangle = \left(\Gamma(B_{s,\mathrm{H}} \to f) + \Gamma(B_{s,\mathrm{L}} \to f)\right) e^{-t/\tau_{B_s}} \left\{ \cosh\left(y_s\, t/\tau_{B_s}\right) + \mathcal{A}^f_{\Delta\Gamma} \sinh\left(y_s\, t/\tau_{B_s}\right) \right\}$$

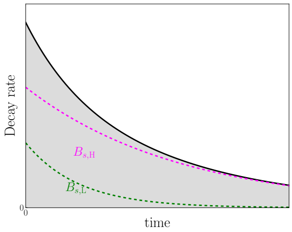
$$\boxed{\mathcal{A}^f_{\Delta\Gamma} = \frac{\Gamma(B_{s,\mathrm{H}} \to f) - \Gamma(B_{s,\mathrm{L}} \to f)}{\Gamma(B_{s,\mathrm{H}} \to f) + \Gamma(B_{s,\mathrm{L}} \to f)}} \qquad \left(\text{a.k.a } D_f = \frac{2\,\mathrm{Re}\,\lambda_f}{1 + |\lambda_f|^2}\right)$$

Time-integrated untagged rate

Experiment measures:

$$\overline{\mathrm{BR}}(B_s \to f) \equiv \frac{1}{2}\int \langle \Gamma(B_s(t) \to f\rangle \, dt$$

$$= \frac{1}{2}\left[\frac{\Gamma(B_{s,\mathrm{H}} \to f)}{\Gamma_\mathrm{H}} + \frac{\Gamma(B_{s,\mathrm{L}} \to f)}{\Gamma_\mathrm{L}}\right]$$



Theoretical calculation in flavor basis:

$$\Gamma(B_s^0 \to f) + \Gamma(\bar{B}_s^0 \to f) = \langle \Gamma(B_s(t) \to f)\rangle\big|_{t=0}$$

$$\mathrm{BR}(B_s \to f) \equiv \frac{\langle \Gamma(B_s(t) \to f)\rangle\big|_{t=0}}{\frac{1}{2}(\Gamma_\mathrm{H} + \Gamma_\mathrm{L})} = \frac{1}{2}\left[\frac{\Gamma(B_{s,\mathrm{H}} \to f)}{\frac{1}{2}(\Gamma_\mathrm{H} + \Gamma_\mathrm{L})} + \frac{\Gamma(B_{s,\mathrm{L}} \to f)}{\frac{1}{2}(\Gamma_\mathrm{H} + \Gamma_\mathrm{L})}\right]$$

Dictionary : $$\overline{\mathrm{BR}}(B_s \to f) = \mathrm{BR}(B_s \to f)\left[\frac{1 + y_s \mathcal{A}^f_{\Delta\Gamma}}{1 - y_s^2}\right]$$

K. Bruyn, R. Fleischer, RK, P. Koppenburg, M. Merk, N. Tuning, Phys.Rev.D 86 (2012)

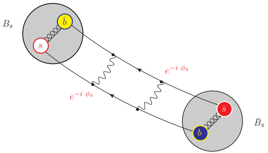
$\mathcal{A}^f_{\Delta\Gamma}$: mass eigenstate rate asymmetry

- Final state dependent, sensitive to New Physics

Consider: $B_s \to f$ with $\mathcal{CP}|f\rangle = \eta_f |f\rangle$

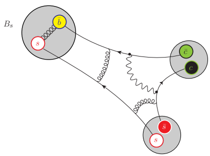
B_s^0–$\bar{B}_s^0$ Mixing:

ϕ_s



Decay Mode:

$\Delta\phi_f$, C_f (direct CPV)



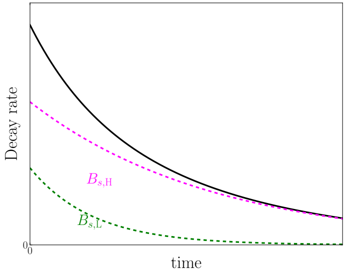
Tagged CP observable : $S_f = \eta_f \sqrt{1 - C_f^2}\, \sin(\phi_s + \Delta\phi_f)$

$$\mathcal{A}^f_{\Delta\Gamma} = -\eta_f \sqrt{1 - C_f^2}\, \cos(\phi_s + \Delta\phi_f)$$

Probe NP with **untagged** time-dependent measurements?

The Effective Lifetime

Approximate untagged rate $\langle \Gamma_f \rangle$ with single exponential $\frac{1}{\tau_f} e^{-t/\tau_f}$



Maximum likelihood fit gives:

$$\tau_f = \frac{\int_0^\infty t \, \langle \Gamma_f \rangle \, dt}{\int_0^\infty \langle \Gamma_f \rangle \, dt}$$

$$= \frac{\tau_{B_s}}{1 - {y_s}^2} \left(\frac{1 + 2\, \mathcal{A}^f_{\Delta\Gamma}\, y_s + {y_s}^2}{1 + \mathcal{A}^f_{\Delta\Gamma}\, y_s} \right)$$

Recover theoretical BR:

$$\mathrm{BR}(B_s \to f) = \underbrace{\left[2 - (1 - {y_s}^2) \frac{\tau_f}{\tau_{B_s}} \right] \overline{\mathrm{BR}}(B_s \to f)}_{\text{all measurable quantities}}$$

Lifetime contours in the ϕ_s–$\Delta\Gamma_s$ plane

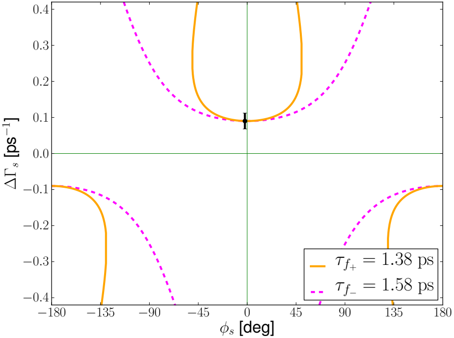
$$\mathcal{CP}|f\rangle = \eta_f |f\rangle \quad \Longrightarrow \quad \mathcal{A}^f_{\Delta\Gamma} = -\eta_f \sqrt{1 - C_f^2}\cos(\phi_s + \Delta\phi_f)$$

$$\tau_f = \frac{\tau_{B_s}}{1 - {y_s}^2}\left(\frac{1 + 2\,\mathcal{A}^f_{\Delta\Gamma}\,y_s + {y_s}^2}{1 + \mathcal{A}^f_{\Delta\Gamma}\,y_s}\right) = \mathrm{function}(\Delta\Gamma_s,\ \phi_s + \Delta\phi_f,\ C_f)$$

Assuming:

$$\Delta\phi_f = 0, \ \ C_f = 0$$

$$\mathcal{A}^f_{\Delta\Gamma} = \begin{cases} -\cos\phi_s & : \quad f_{\mathrm{even}} \\ +\cos\phi_s & : \quad f_{\mathrm{odd}} \end{cases}$$



Lifetime contours in the ϕ_s–$\Delta\Gamma_s$ plane

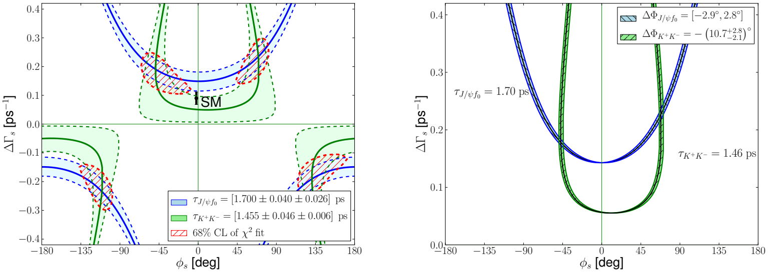
$$\tau(B_s \to f) = \text{function}\left(\Delta\Gamma_s,\ \boxed{\phi_s + \Delta\phi_f},\ C_f\right)$$

- $\boxed{\textbf{CP Even}:\ \tau(B_s \to K^+K^-)}$ $\Delta\phi_{K^+K^-} = -\left(10.5^{+3.1}_{-2.8}\right)^\circ,\quad C_{K^+K^-} = 0.09$

LHCb, Phys.Lett. B716 (2012) 393-400; R. Fleischer, RK, Eur.Phys.J. C71 (2011) 1532

- $\boxed{\textbf{CP Odd}:\ \tau(B_s \to J/\psi\, f_0(980))}$ $\Delta\phi_{J/\psi\, f_0} \in [-3^\circ, 3^\circ],\quad C_{J/\psi\, f_0} \le 0.05$

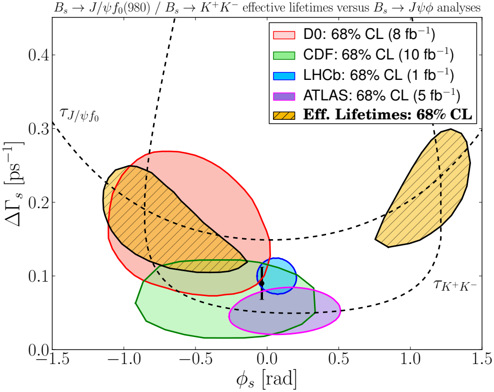
LHCb, Phys.Rev.Lett. 109 (2012) 152002; R. Fleischer, RK, G. Ricciardi, Eur.Phys.J. C71 (2011) 1832

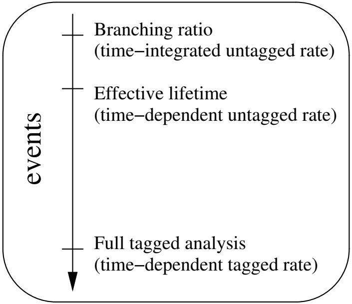


R. Fleischer and RK, Eur.Phys.J. C71 (2011) 1532; RK, C12-06-11.2, arXiv:1209.3206

Comparison with tagged measurements

Full tagged $B_s \to J/\psi\phi$ analysis:





Upcoming *untagged time-dependent* measurements?

The Rare Decay $B_s \to \mu^+\mu^-$

$$\overline{\mathrm{BR}}(B_s \to \mu^+\mu^-) = \begin{cases} \left(2.9^{+1.1}_{-1.0}\right) \times 10^{-9} & \mathrm{LHCb} \\ \left(3.0^{+1.0}_{-0.9}\right) \times 10^{-9} & \mathrm{CMS} \end{cases} = (2.9 \pm 0.7) \times 10^{-9}$$

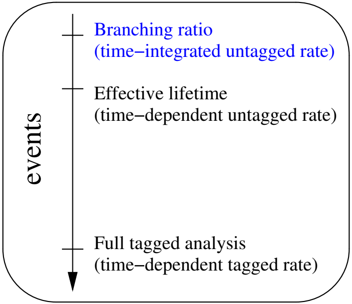
LHCb: Phys.Rev.Lett. 111 (2013) 101805, CMS: Phys.Rev.Lett. 111 (2013) 101804

- Standard Model: **only** $B_{s,\mathrm{H}} \to \mu^+\mu^-$
- Including y_s effects with $\mathcal{A}^{\mu\mu}_{\Delta\Gamma}|_{\mathrm{SM}} = 1$:

$$\overline{\mathrm{BR}}(B_s \to \mu^+\mu^-)_{\mathrm{SM}} = (3.56 \pm 0.18)\times 10^{-9}$$

A.J. Buras, J. Girrbach, D. Guadagnoli, G. Isidori, Eur.Phys.J. C72 (2012) 2172

A.J. Buras, R. Fleischer, J. Girrbach, RK, JHEP 1307 (2013) 77

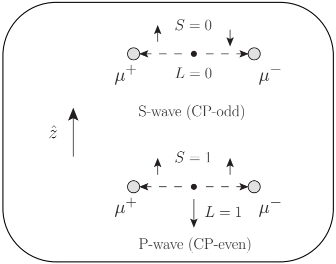


$$\overline{R} \equiv \frac{\overline{\mathrm{BR}}(B_s \to \mu^+\mu^-)}{\overline{\mathrm{BR}}(B_s \to \mu^+\mu^-)_{\mathrm{SM}}}, \qquad \boxed{\overline{R}_{\mathrm{LHCb}} = 0.83 \pm 0.20, \qquad \overline{R}_{\mathrm{SM}} = 1}$$

K. Bruyn, R. Fleischer, RK, P. Koppenburg, M. Merk, A. Pellegrino, N. Tuning, Phys.Rev.Lett 109 (2012)

$B_s \to \mu^+\mu^-$ beyond the Standard Model

$$\mathcal{H}_{\rm eff} \propto \sum_i^{\{10,S,P\}} C_i\,\mathcal{O}_i + C_i'\,\mathcal{O}_i'$$



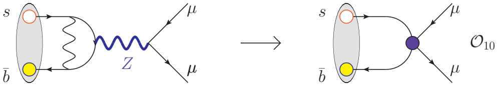
$$\boldsymbol{\mathcal{O}_{10} = (\bar{s}\gamma_\mu P_L b)(\bar{\mu}\gamma^\mu\gamma_5\mu)}$$

$$\mathcal{O}_{\mathcal{S}} = (\bar{s}P_R b)(\bar{\mu}\mu)$$

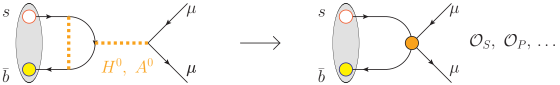
$$\mathcal{O}_{\mathcal{P}} = (\bar{s}P_R b)(\bar{\mu}\gamma_5\mu)$$

(and $P_L \leftrightarrow P_R$ for $\mathcal{O}'$)

Standard Model: only $\boldsymbol{\mathcal{O}_{10}} \implies$ only $\boldsymbol{B_{s,\mathrm{H}}} \to \mu^+\mu^-$



Beyond the SM: non-vanishing $\boldsymbol{B_{s,\mathrm{L}}} \to \mu^+\mu^-$?



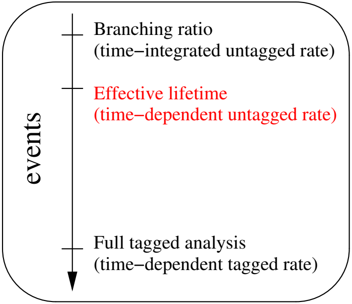
$B_s \to \mu^+\mu^-$ time-dependent measurement

Define for convenience:

$$\boldsymbol{P} \equiv \frac{C_{10} - C_{10}'}{C_{10}^{\rm SM}} + \frac{m_{B_s}^2}{2\,m_\mu}\left(\frac{m_b}{m_b + m_s}\right)\left(\frac{C_P - C_P'}{C_{10}^{\rm SM}}\right)$$

$$\boldsymbol{S} \equiv \sqrt{1 - \frac{4\,m_\mu^2}{m_{B_s}^2}}\,\frac{m_{B_s}^2}{2\,m_\mu}\left(\frac{m_b}{m_b + m_s}\right)\left(\frac{C_S - C_S'}{C_{10}^{\rm SM}}\right)$$

In SM: $\boldsymbol{P} \to 1$, $\boldsymbol{S} \to 0$



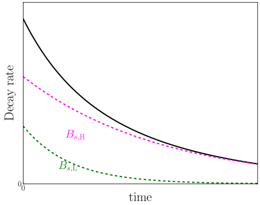
$$\Gamma(\boldsymbol{B_{s,\mathrm{L}}} \to \mu^+\mu^-) \propto |\boldsymbol{P}|^2 \underbrace{\sin^2(\varphi_{\boldsymbol{P}} - \phi_s^{\rm NP}/2)}_{\text{new CP phases}} + \underbrace{|\boldsymbol{S}|^2 \cos^2(\varphi_{\boldsymbol{S}} - \phi_s^{\rm NP}/2)}_{\text{scalar operators}}$$

Probe **NP** with :

$$\mathcal{A}_{\Delta\Gamma}^{\mu\mu} \equiv \frac{\Gamma(B_{s,\mathrm{H}} \to \mu^+\mu^-) - \Gamma(B_{s,\mathrm{L}} \to \mu^+\mu^-)}{\Gamma(B_{s,\mathrm{H}} \to \mu^+\mu^-) + \Gamma(B_{s,\mathrm{L}} \to \mu^+\mu^-)}$$

e.g. with $B_s \to \mu^+\mu^-$ Effective Lifetime

$B_s \to \mu^+\mu^-$ untagged observables



$$\overline{R} = \frac{(1 + y_s\, \mathcal{A}_{\Delta\Gamma}^{\mu\mu})}{1 + y_s} \left(|P|^2 + |S|^2\right)$$

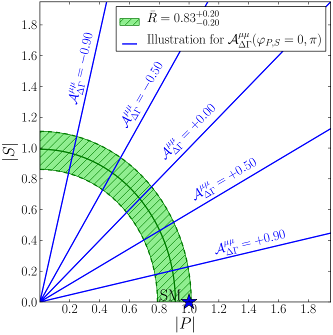
$$\mathcal{A}_{\Delta\Gamma}^{\mu\mu} = \frac{|P|^2 \cos\left(2\varphi_P - \phi_s^{\mathrm{NP}}\right) - |S|^2 \cos\left(2\varphi_S - \phi_s^{\mathrm{NP}}\right)}{|P|^2 + |S|^2}$$

K. Bruyn, R. Fleischer, RK, P. Koppenburg, M. Merk, A. Pellegrino, N. Tuning, Phys.Rev.Lett 109 (2012)

Solvable scenarios:

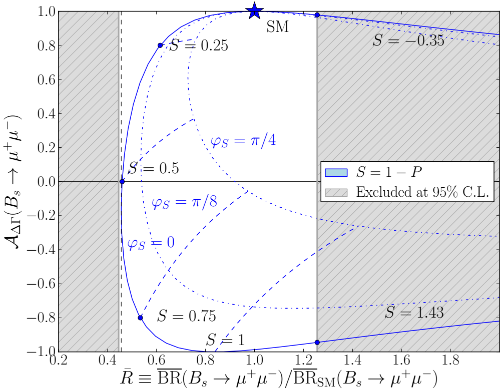
- A: $|P|$, φ_P free ($S = 0$)
- B: $|S|$, φ_S free ($P = 1$)
- C: $S = \pm[1 - P]$
- D: $\varphi_P = \varphi_S = 0$: $\longrightarrow$

A.J. Buras, R. Fleischer, J. Girrbach, RK, JHEP 1307 (2013) 77



New scalar and pseudoscalar operators on same footing

Scenario C: $P = 1 + \tilde{P},\ \boldsymbol{S = \pm\tilde{P}}$

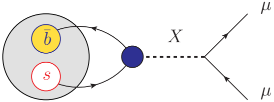


Realised for:

$$\boxed{C_S^{(\prime)} = \pm C_P^{(\prime)}}$$

E.g: Decoupled 2HDM/MSSM ($M_{H^0} \approx M_{A^0} \gg M_{h^0}$)

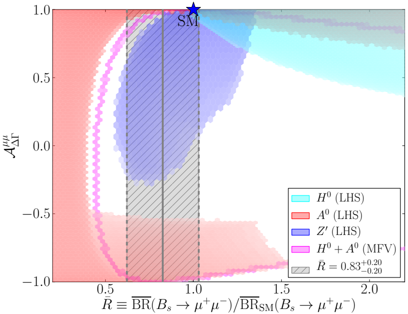
Specific models in the $\overline{R}$-$\mathcal{A}_{\Delta\Gamma}^{\mu\mu}$ parameter space



$X \in \{ \quad \boldsymbol{Z'}, \quad \boldsymbol{H^0}, \quad \boldsymbol{A^0}, \quad \boldsymbol{H^0 + A^0} \quad \},$

$$M_X = 1\,\mathrm{TeV}$$

- B_s mixing (ΔM_s, ϕ_s) constrains quark couplings
- Lepton couplings left free



A.J. Buras, R. Fleischer, J. Girrbach, RK, JHEP 1307 (2013) 77

A.J. Buras, F. De Fazio, J. Girrbach, RK, M. Nagai, JHEP 1306 (2013) 111

Very possible that **NP hidden** within $\overline{\mathrm{BR}}(B_s \to \mu\mu)$ bounds.
Expose with time-dependent measurement!

$B_s \to \mu^+\mu^-$ tagged analysis

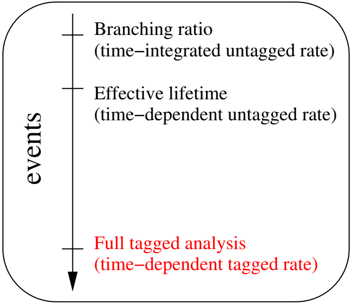
Eventually also tagged measurement:

$$\frac{\Gamma(B_s^0(t)\to\mu^+\mu^-)-\Gamma(\bar{B}_s^0(t)\to\mu^+\mu^-)}{\Gamma(B_s^0(t)\to\mu^+\mu^-)+\Gamma(\bar{B}_s^0(t)\to\mu^+\mu^-)} = \frac{\boldsymbol{\mathcal{S}_{\mu\mu}}\sin(\Delta M_s t)}{\cosh(y_s t/\tau_{B_s}) + \boldsymbol{\mathcal{A}^{\mu\mu}_{\Delta\Gamma}}\sinh(y_s t/\tau_{B_s})}.$$

- $\mathcal{S}_{\mu\mu}$ **indepedent** if scalar operators:

$$|\boldsymbol{\mathcal{A}^{\mu\mu}_{\Delta\Gamma}}|^2 + |\boldsymbol{\mathcal{S}_{\mu\mu}}|^2 = 1 - \left[\frac{2|P||S|\cos(\varphi_P-\varphi_S)}{|P|^2+|S|^2}\right]^2$$

- $\mathcal{S}_{\mu\mu}$ sensitive to small CP phases



Summary

$\boxed{\Gamma_{\rm L} \neq \Gamma_{\rm H}}$ implies:

- Mass-eigenstate rate asymmetries $\left(\mathcal{A}^f_{\Delta\Gamma}\right)$ from **time-dependent untagged** measurements

  e.g. **effective lifetimes**

- Branching ratio dictionary:

$$\mathrm{BR}(B_s \to f) \quad \overset{\mathcal{A}^f_{\Delta\Gamma}/\tau_f}{\longleftrightarrow} \quad \overline{\mathrm{BR}}(B_s \to f)$$

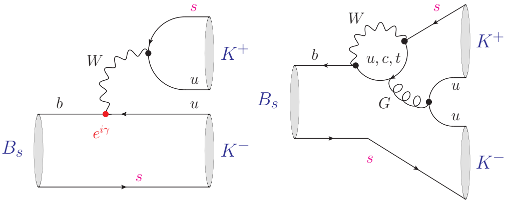
- B_s^0–$\bar{B}_s^0$ mixing constraints from effective lifetimes $(\tau_{f_+},\ \tau_{f_-})$
- New topic for LHC upgrade:

  $\boxed{\text{effective lifetime for } B_s \to \mu^+\mu^-}$

Backup Slides

Controlling the **CP Even** Decay Mode

$$B_s \to K^+K^-$$



$$\Delta\phi_{K^+K^-} = -\left(10.5^{+3.1}_{-2.8}\right)^\circ$$

$$C_{K^+K^-} = 0.09 \pm 0.05$$

- Use *U*-**spin** flavour symmetry (subgroup $SU(3)_F$):

R. Fleischer, Phys.Lett.B 459 (1999) 306

interchange $s \leftrightarrow d$ quarks

Related to $B_d \to \pi^+\pi^-$

Extract **CP violating phase**:

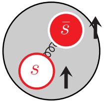
$$\gamma = \left(68^{+5}_{-6}\Big|_{\text{input}}{}^{+5}_{-4}\Big|_{\text{U-spin}}\right)^\circ = (68 \pm 7)^\circ$$

R. Fleischer and RK, Eur.Phys.J. C71 (2011) 1532

Controlling the **CP Odd** Decay Mode

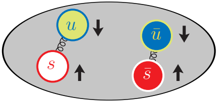
$$B_s \to J/\psi\, f_0(980)$$

Quark-antiquark

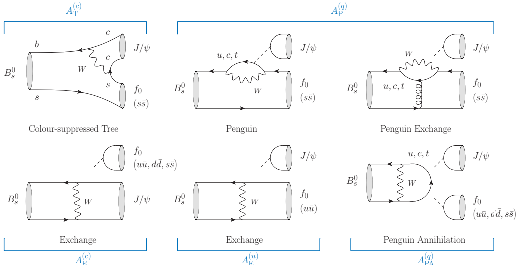


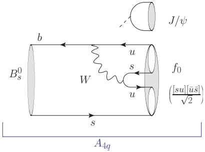
What is $f_0(980)$?

Tetraquark



- Decay amplitudes may vary:





Controlling the **CP Odd** Decay Mode

- With SM CP violation and **unknown decay amplitudes**:

$$\boxed{\Delta\phi_{J/\psi f_0} \in [-3^\circ, 3^\circ]}, \quad C_{J/\psi f_0} \lesssim 0.05$$

- Proposed **control channel**: $B_d \to J/\psi\, f_0(980)$

Predict : $\mathrm{BR}(B_d \to J/\psi\, f_0; f_0 \to \pi^+\pi^-)$

$$\sim (1-3) \times 10^{-6} \times \begin{cases} \left[\frac{\tan\varphi_M}{\tan 35^\circ}\right]^2 & : \quad q\bar{q} \\ 1 & : \quad \text{tetraquark} \end{cases}$$

R. Fleischer, RK, G. Ricciardi, Eur.Phys.J. C71 (2011) 1832

So far:

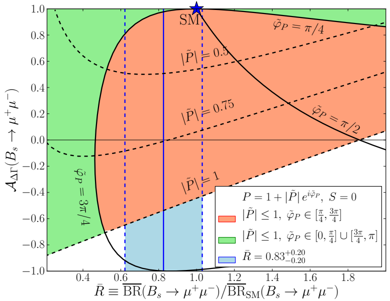
$$\mathrm{BR}(B_d \to J/\psi\, f_0; f_0 \to \pi^+\pi^-)_{\mathrm{LHCb}} < 1.1 \times 10^{-6} \ (90\% \ \mathrm{C.L})$$

LHCb, Phys. Rev. D87 (2013) 052001

No scalar operators OR only new scalar operators

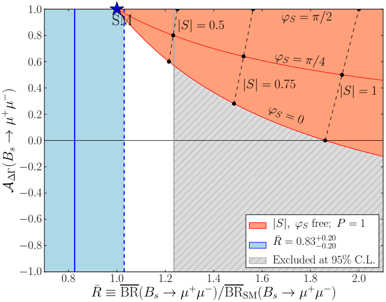
$$\Gamma(B_{s,\mathrm{L}} \to \mu^+\mu^-) \propto |P|^2 \underbrace{\sin^2(\varphi_P - \phi_s^{\mathrm{NP}}/2)}_{\text{scenario A}} + \underbrace{|S|^2 \cos^2(\varphi_S - \phi_s^{\mathrm{NP}}/2)}_{\text{scenario B}}$$

Scenario A: $S = 0$



E.g: CMFV, Z' Models, A^0 dominant (2HDM)

Scenario B: $S \neq 0$ ($P = 1$)



E.g: H^0 dominant (2HDM)

$B_s \to \mu^+\mu^-$ tagged analysis

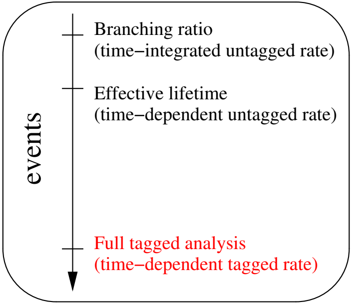
Eventually also tagged measurement:

$$\frac{\Gamma(B_s^0(t)\to\mu^+\mu^-)-\Gamma(\bar{B}_s^0(t)\to\mu^+\mu^-)}{\Gamma(B_s^0(t)\to\mu^+\mu^-)+\Gamma(\bar{B}_s^0(t)\to\mu^+\mu^-)} = \frac{\boldsymbol{\mathcal{S}_{\mu\mu}}\sin(\Delta M_s t)}{\cosh(y_s t/\tau_{B_s})+\boldsymbol{\mathcal{A}^{\mu\mu}_{\Delta\Gamma}}\sinh(y_s t/\tau_{B_s})}.$$

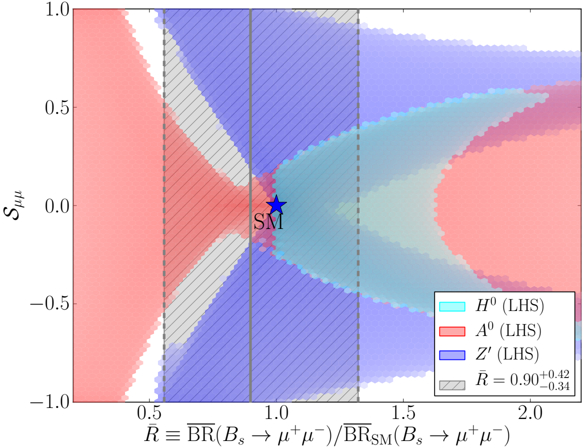
- $\mathcal{S}_{\mu\mu}$ **indepedent** if scalar operators:

$$|\boldsymbol{\mathcal{A}^{\mu\mu}_{\Delta\Gamma}}|^2+|\boldsymbol{\mathcal{S}_{\mu\mu}}|^2 = 1-\left[\frac{2|P||S|\cos(\varphi_P-\varphi_S)}{|P|^2+|S|^2}\right]^2$$

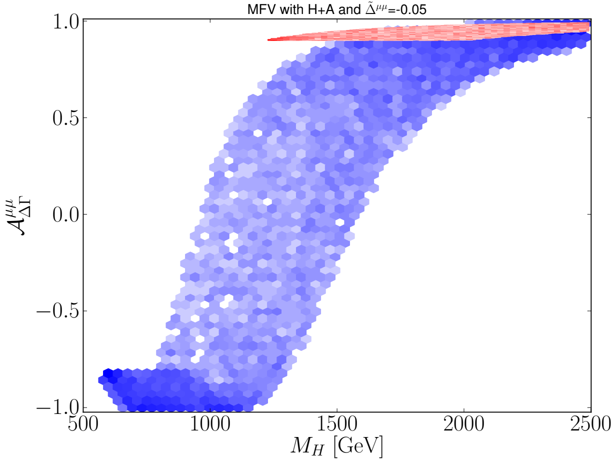
- $\mathcal{S}_{\mu\mu}$ sensitive to small CP phases



Parameter space with $B_s \to \mu^+\mu^-$ tagged analysis



Mass dependence of $\mathcal{A}^f_{\Delta\Gamma}$ in $H^0 + A^0$ model



Fitting an effective lifetime

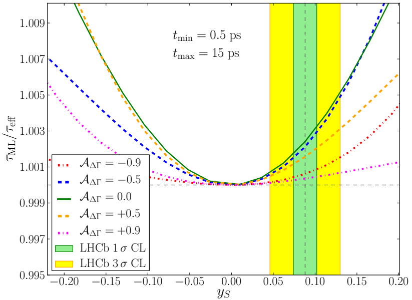
$$f_{\text{true}}(t) \equiv \frac{A(t)\,\langle\Gamma(t)\rangle}{\int_0^\infty A(t)\,\langle\Gamma(t)\rangle\,dt}, \qquad f_{\text{fit}}(t;\tau) \equiv \frac{A(t)\,e^{-t/\tau}}{\int_0^\infty A(t)\,e^{-t/\tau}\,dt}$$

Minimise : $\quad -\log L(\tau) = -n\int_0^\infty dt\;\, f_{\text{true}}(t)\log\left[f_{\text{fit}}(t;\tau)\right]$

$$\frac{\int_0^\infty t\,A(t)e^{-t/\tau}\,dt}{\int_0^\infty A(t)e^{-t/\tau}\,dt} = \frac{\int_0^\infty t\,A(t)\langle\Gamma(t)\rangle\,dt}{\int_0^\infty A(t)\langle\Gamma(t)\rangle\,dt}$$

Limit that $A(t) = 1$:

$$\tau = \frac{\int_0^\infty t\,\langle\Gamma(t)\rangle\,dt}{\int_0^\infty \langle\Gamma(t)\rangle\,dt} \equiv \tau_{\text{eff}},$$



Tagged analysis

The CP asymmetry:

$$\frac{\Gamma\left(B_s(t)\to f\right)-\Gamma(\bar{B}_s(t)\to f)}{\Gamma(B_s(t)\to f)+\Gamma(\bar{B}_s(t)\to f)}=\frac{C\cos(\Delta M_s\,t)+S\,\sin(\Delta M_s\,t)}{\cosh(\Delta\Gamma_s\,t)+\mathcal{A}_{\Delta\Gamma}\sinh(\Delta\Gamma_s\,t)}$$

Observables for $\mathcal{CP}|f\rangle=\eta\,|f\rangle$:

$$\lambda_f\equiv\frac{q}{p}\,\frac{A(\overline{B}_s^0\to f)}{A(B_s^0\to f)}=-\eta\,e^{-i\phi_s}\,\sqrt{\frac{1-C}{1+C}}\,e^{-i\Delta\phi}$$

$$\mathcal{A}_{\Delta\Gamma}+i\,S=\frac{2\,\lambda_f}{1+|\lambda_f|^2}=\boxed{-\eta\,\sqrt{1-C^2}\,e^{-i(\phi_s+\Delta\phi)}}$$